

Optimal two-level choice designs for the main effects and specified interaction effects model

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Abstract

Choice designs for the main effects model, broader main effects model and main effects plus specified interaction effects model are discussed in this paper. Universally optimal choice designs are obtained for all of these models using Hadamard matrix and other combinatorial techniques. Choice experiments under the multinomial logit model for equally attractive options are assumed for finding universally optimal choice designs.

Keywords: Optimal choice designs, Hadamard matrix, Main effects, Interaction effects.

1. Introduction

Discrete choice experiments are widely used for quantifying consumer preferences in various areas including marketing, transport, environmental resource economics and public welfare analysis. A choice experiment consists of a number of choice sets, each containing several options (alternatives, profiles or treatment combinations). Respondents are shown each choice set in turn and are asked which option they prefer from each of the choice sets presented. Each option in a choice set is described by the level combination of n factors (attributes). We assume that there are no repeated options in a choice set and each respondent chooses the best option from each choice set as per their perceived utility. A choice design is a collection of choice sets employed in a choice experiment. Thus a choice design d consisting of N choice sets, each containing m profiles and each profile is a level combination of n factors.

Street and Burgess (2007) present a comprehensive exposition of designs for choice experiments under the multinomial logit model. More recently, Großmann and Schwabe (2015) present a review of choice experiments till date. The literature so far on this subject is mainly focused on optimal choice designs for the main effects model (Burgess and Street (2003); Graßhoff et al. (2004); Burgess and Street (2005); Demirkale et al. (2013) etc.) and for the main effects plus two factor interaction effects model (Burgess and Street (2003); Graßhoff et al. (2003); Burgess and Street (2005); Großmann et al. (2012) etc.). Though for the former case, researchers are able to find optimal designs in very less number of choice sets but in the later case, the proposed optimal designs take a large number of choice sets. For this reason, in most of the cases, the theoretical optimal designs, for estimating all the main effects and two factor interaction effects, are quite impractical to use. To overcome this situation, researchers propose near-optimal designs with relatively lesser number of choice sets by sacrificing the efficiency (Street and Burgess (2004); Street et al. (2005); Street and Burgess (2007) etc.).

Though the advancement of this subject is really splendid in the last decade but there many areas of investigation still remain to explore. In this paper we explore some of them which have some theoretical and practical importance. For example, suppose a researcher

is interested to estimate all the main effects but (s)he can not deny the presence of two factor interaction effects in the model. So, in this situation, (s)he wants an optimal design for the estimation of main effects in the presence of two factor interaction effects but in the absence of three and higher order interaction effects in the model. We refer this model as broader main effects model and obtain optimal designs under this model. In most of the choice situations, the information of all the main effects and all the two factor interactions are not so important but the information about all the main effects and two or higher order interaction effects with some specific factors are much more important to investigate. In this paper optimal choice designs are obtained for such situations in practical number of choice sets. It gives more flexibility to the researchers to design their own choice experiment appropriately for estimating all the main effects and specified two or higher order interaction effects of their interest.

In this paper we restrict ourselves in 2^n choice experiments under the multinomial logit model. A general set up and characterization of information matrix is given in Section 2. Optimal choice designs for the main effects under the main effects model and the broader main effects model are obtained in Section 3. In Section 4, optimal choice designs are obtained for the main effects and specified interaction effects (with one factor or more than one factors) model. Finally, In Section 5, a general discussion is given on achieved designs.

2. Useful notations and information matrix

Let $d = d_{N,n,m} = \{(T_1, \dots, T_m)\}$ is a choice design with N choice sets each of size m with n factors, where a typical treatment combination $T_i = (i_1 \dots i_r \dots i_n)$, $i_r = 0, 1; r = 1, \dots, n$. Let A_i , $i = 1, \dots, m$, be $N \times n$ matrices with entries 0 and 1. Then a choice design d can also be represented in matrix notation as $d = (A_1, \dots, A_m)$, where the p -th row from each A_i makes the p -th choice set S_p (say) and hence $d = \{S_p : p = 1, \dots, N\}$.

Let f_1, \dots, f_n , denote the n factors and let $F_{h_1 \dots h_r}$ denotes the r -th order interaction effect corresponding to the factors f_{h_1}, \dots, f_{h_r} . Clearly, when $r = 1$, F_{h_1} denotes the main effect of the factor f_{h_1} and when $r = 2$, $F_{h_1 h_2}$ denotes the two factor interaction effect between the factors f_{h_1} and f_{h_2} and so on. We define the position of the factor f_{h_r} in a treatment T_i as i_{h_r} and the effective position of a factorial effect $F_{h_1 \dots h_r}$ in a treatment T_i as $i_{h_1 \dots h_r}^* = r + 1 - (i_{h_1} + \dots + i_{h_r}) \pmod{2}$. Let $S_p(h_1 \dots h_r) = (1_{h_1 \dots h_r}^*, \dots, m_{h_1 \dots h_r}^*)_{h_1 \dots h_r}$ be the effective choice set of $S_p = (T_1, \dots, T_m)$ for the factorial effect $F_{h_1 \dots h_r}$. Similarly, let $S_p(h_1 \dots h_r, k_1 \dots k_l) = (1_{h_1 \dots h_r, k_1 \dots k_l}^*, \dots, m_{h_1 \dots h_r, k_1 \dots k_l}^*)_{h_1 \dots h_r, k_1 \dots k_l}$ be the effective choice set of $S_p = (T_1, \dots, T_m)$ for any two factorial effects $F_{h_1 \dots h_r}$ and $F_{k_1 \dots k_l}$.

In this paper we use the standard definition of contrast vector corresponding to a factorial effect $F_{h_1 \dots h_r}$. Let $B_{h_u} = (b_{h_u}^{(j)})$ be the orthogonal contrast vector of the factorial effect F_{h_u} , $h_u = 1, \dots, n$. Corresponding to a treatment T_i and the factorial effect F_{h_u} , let $b_{h_u}^{(i)} = -1$ if $i_{h_u} = 0$ and $b_{h_u}^{(i)} = 1$ if $i_{h_u} = 1$. Let $B_{h_1 \dots h_r} = (b_{h_1 \dots h_r}^{(j)})$ be the orthogonal contrast vector of the factorial effect $F_{h_1 \dots h_r}$. Then corresponding to a treatment T_i , $b_{h_1 \dots h_r}^{(i)} = b_{h_1}^{(i)} \dots b_{h_r}^{(i)}$. It is assumed that the treatments are arranged in lexicographic order in $B_{h_1 \dots h_r}$.

Let Λ be the information matrix of treatment effects corresponding to a design d , and let B be the orthogonal treatment contrast matrix corresponding to all the factorial effects of interest. Then the information matrix of the factorial effects of interest corresponding to d is $C_d = (1/2^n)B\Lambda B'$ (see, street 2007 for details). A design d is connected if the corresponding information matrix C_d ($= C$ say) is positive definite. A connected design allows the estimation of all underlying factorial effects of interest.

Let $\mathcal{D}_{N,n,m}$ be the class of all connected designs with N choice sets each of size m with n factors. Note from (street 2007) that for a design $d \in \mathcal{D}_{N,n,m}$, the $2^n \times 2^n$ information matrix $\Lambda = ((\lambda_{st}))$ of the treatment effects for equally attractive options is

$$\lambda_{st} = \begin{cases} ((m-1)/Nm^2) \sum_{j_2 < \dots < j_m} N_{j_1 j_2 \dots j_m} & \text{if } s = t = j_1 \\ (-1/Nm^2) \sum_{j_3 < \dots < j_m} N_{j_1 j_2 \dots j_m} & \text{if } s = j_1, t = j_2 \\ 0 & \text{otherwise,} \end{cases}$$

where $N_{j_1 \dots j_m}$ is the indicator function taking value 1 if $(T_{j_1}, \dots, T_{j_m}) \in d$ and 0 otherwise. Let $M^{(j_1 \dots j_m)} = ((m_{st}))$ be a $2^n \times 2^n$ matrix corresponding to a choice set $(T_{j_1}, \dots, T_{j_m})$, where

$$m_{st} = \begin{cases} m-1 & \text{if } s = t, t \in \{j_1, \dots, j_m\} \\ -1 & \text{if } s \neq t, (s, t) \in \{j_1, \dots, j_m\} \\ 0 & \text{otherwise.} \end{cases}$$

Then for any choice design d in $\mathcal{D}_{N,n,m}$, Λ can be written as

$$\Lambda = (1/Nm^2) \sum_{j_1 < \dots < j_m} N_{j_1 \dots j_m} M^{(j_1 \dots j_m)} = (1/Nm^2) \Lambda^* \text{ (say).}$$

We consider the matrix $M^{(j_1 \dots j_m)}$ as the contribution of the choice set $(T_{j_1}, \dots, T_{j_m})$ to Λ . The definition of $M^{(j_1 \dots j_m)}$ suggests that we can write $M^{(j_1 \dots j_m)} = \sum_{j_r < j_{r'}} M^{(j_r j_{r'})}$, where

$j_r, j_{r'} \in \{j_1, \dots, j_m\}$. Thus the contribution of the choice set $(T_{j_1}, \dots, T_{j_m})$ to Λ is the sum of the contributions of all its $m(m-1)/2$ component pairs $(T_{j_r}, T_{j_{r'}})$. Then the information matrix $C = ((c_{qq'}))$, for the factorial effects of interest can be written as

$$\begin{aligned} C &= (1/2^n) BAB' = (1/2^n Nm^2) B \Lambda^* B' \\ &= (1/2^n Nm^2) \sum_{j_1 < \dots < j_m} N_{j_1 \dots j_m} \left\{ B \left(\sum_{j_r < j_{r'}} M^{(j_r j_{r'})} \right) B' \right\}. \end{aligned} \quad (2.1)$$

Since each choice set S_p contains $m(m-1)/2$ component pairs (T_i, T_j) and there are N such choice sets in d , therefore the total number of component pairs in a design d is $N^* = Nm(m-1)/2$. If $B_{h_1 \dots h_r}$ and $B_{k_1 \dots k_l}$ are the q -th and q' -th contrasts B , then from the expression (2.1) we see that the values of $c_{qq'}$ are only depend on the values of $B_{h_1 \dots h_r} M^{(ij)} B'_{k_1 \dots k_l}$, for all the N^* component pairs (T_i, T_j) in d . The following result helps us to determine the values of $c_{qq'}$.

Lemma 2.1. *For a component pair (T_i, T_j) , the exhaustive cases indicating possible values of $B_{h_1 \dots h_r} M^{(ij)} B'_{k_1 \dots k_l}$ are*

Case 1: $B_{h_1 \dots h_r} M^{(ij)} B'_{k_1 \dots k_l} = 4$ when $(i_{h_1 \dots h_r}^ i_{k_1 \dots k_l}^*, j_{h_1 \dots h_r}^* j_{k_1 \dots k_l}^*)_{h_1 \dots h_r, k_1 \dots k_l} = (00, 11)_{h_1 \dots h_r, k_1 \dots k_l}$*

Case 2: $B_{h_1 \dots h_r} M^{(ij)} B'_{k_1 \dots k_l} = -4$ when $(i_{h_1 \dots h_r}^ i_{k_1 \dots k_l}^*, j_{h_1 \dots h_r}^* j_{k_1 \dots k_l}^*)_{h_1 \dots h_r, k_1 \dots k_l} = (01, 10)_{h_1 \dots h_r, k_1 \dots k_l}$*

Case 3: $B_{h_1 \dots h_r} M^{(ij)} B'_{k_1 \dots k_l} = 0$ for all other situations.

Proof. Let $B_x = (x_1, \dots, x_i, \dots, x_j, \dots, x_{2^n})$ and $B_y = (y_1, \dots, y_i, \dots, y_j, \dots, y_{2^n})$ are the contrast vectors corresponding to $F_{h_1 \dots h_r}$ and $F_{k_1 \dots k_l}$ respectively. Note that $M^{(ij)}$ is a $2^n \times 2^n$ matrix with all elements 0 except $M_{ii}^{(ij)} = M_{jj}^{(ij)} = 1$ and $M_{ij}^{(ij)} = M_{ji}^{(ij)} = -1$. Therefore $B_x M^{(ij)} B'_y = (0, \dots, (x_i - x_j), \dots, -(x_i - x_j), \dots, 0) B'_y = (x_i - x_j)y_i - (x_i - x_j)y_j = (x_i - x_j)(y_i - y_j)$.

The results then simply follows from the definitions of $B_{h_1 \dots h_r}$ and $B_{k_1 \dots k_l}$. \square

Corollary 2.1. *For a component pair (T_i, T_j) , $B_{h_1 \dots h_r} M^{(ij)} B'_{h_1 \dots h_r} = 4$ when $(i_{h_1 \dots h_r}^*, j_{h_1 \dots h_r}^*)_{h_1 \dots h_r} = (0, 1)_{h_1 \dots h_r}$, and 0 otherwise.*

Remark 2.1. *From the Lemma 2.1 and Corollary 2.1, we see that for a component pair (T_i, T_j) , the values of $B_{h_1 \dots h_r} M^{(ij)} B'_{k_1 \dots k_l}$ and $B_{h_1 \dots h_r} M^{(ij)} B'_{h_1 \dots h_r}$ are only depends on the effective component pairs $(T_i, T_j)_{h_1 \dots h_r, k_1 \dots k_l} = (i_{h_1 \dots h_r}^*, i_{k_1 \dots k_l}^*, j_{h_1 \dots h_r}^*, j_{k_1 \dots k_l}^*)_{h_1 \dots h_r, k_1 \dots k_l}$ and $(T_i, T_j)_{h_1 \dots h_r} = (i_{h_1 \dots h_r}^*, j_{h_1 \dots h_r}^*)_{h_1 \dots h_r}$ respectively.*

Let F is the set of Q factorial effects $F_{h_1 \dots h_r}$'s of interest corresponding to a choice design d . Then B would be a $Q \times 2^n$ contrast matrix and C would be a square matrix of order Q . Let $N_F = \{h_1 \dots h_r : F_{h_1 \dots h_r} \in F\}$. Now for any two $h_1 \dots h_r, k_1 \dots k_l \in N_F$, we define, $\eta_{h_1 \dots h_r, k_1 \dots k_l}^+$ and $\eta_{h_1 \dots h_r, k_1 \dots k_l}^-$ to be the total number of effective component pairs of the type $(00, 11)_{h_1 \dots h_r, k_1 \dots k_l}$ and $(01, 10)_{h_1 \dots h_r, k_1 \dots k_l}$ respectively in d . We also define, $n_p(h_1 \dots h_r)$ to be the total number of 0's in the effective choice set $S_p(h_1 \dots h_r)$ of d , $p = 1, \dots, N$, $h_1 \dots h_r \in N_F$.

We use the universal optimality criteria for finding optimal designs in \mathcal{D} . Following Kiefer (1975), a choice design d^* is universally optimal in \mathcal{D} , if C_{d^*} is a scalar multiple of identity matrix and $\text{trace}(C_{d^*}) \geq \text{trace}(C_d)$, for any other design $d \in \mathcal{D}$. If a design d is universally optimal in \mathcal{D} , then it is also A -, D -, and E -optimal. We have the following results for diagonal C -matrix and maximum value of $\text{trace}(C)$ for a design d in $\mathcal{D}_{N,n,m}$.

Lemma 2.2. *For $q \neq q'$, $c_{qq'} = 0$, if and only if $\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ = \eta_{h_1 \dots h_r, k_1 \dots k_l}^-$.*

Proof. Let $c_{qq'}^*$ denotes the (q, q') -th element of $B\Lambda^*B'$, $q \neq q'$. Then it follows from the expression (2.1) and from the Lemma 2.1 that

$$\begin{aligned} C_{qq'}^* &= \sum_{j_1 < \dots < j_m} N_{j_1 \dots j_m} \sum_{j_r < j_{r'}} \{B_{h_1 \dots h_r} M^{(j_r j_{r'})} B'_{k_1 \dots k_l}\} \\ &= [4(\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ - \eta_{h_1 \dots h_r, k_1 \dots k_l}^-) + 0\{N^* - (\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ + \eta_{h_1 \dots h_r, k_1 \dots k_l}^-)\}]. \end{aligned}$$

Thus $c_{qq'}^*$ or equivalently $c_{qq'} = 0$, if and only if $\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ = \eta_{h_1 \dots h_r, k_1 \dots k_l}^-$. \square

Lemma 2.3. *Let d be a design in $\mathcal{D}_{N,n,m}$, then*

$$\max(\text{trace}(C)) = \begin{cases} Q/2^n & \text{for } m \text{ even} \\ Q(m^2 - 1)/2^n m^2 & \text{for } m \text{ odd,} \end{cases}$$

and the $\max(\text{trace}(C))$ occurs when $n_p(h_1 \dots h_r) = m/2$ (m even) and $n_p(h_1 \dots h_r) = (m - 1)/2$ or $(m+1)/2$ (m odd) for every effective choice set $S_p(h_1 \dots h_r)$, $p = 1, \dots, N$, $h_1 \dots h_r \in N_F$.

Proof. Let $B_{h_1 \dots h_r}$ be the q -th contrast of B corresponding to the factorial effect $F_{h_1 \dots h_r}$ and let c_{qq}^* be the (q, q) -th element of $B\Lambda^*B'$. Note from (2.1) that every component pair (T_i, T_j) adds a value $B_{h_1 \dots h_r} M^{(ij)} B'_{h_1 \dots h_r}$ to c_{qq}^* . From Corollary 2.1 and Remark 2.1, we see that

this value is 4 if and only if the effective pair $(T_i, T_j)_{h_1 \dots h_r}$ has $i_{h_1 \dots h_r}^* \neq j_{h_1 \dots h_r}^*$. Since the contribution of each effective choice set $S_p(h_1 \dots h_r)$ to c_{qq}^* is equivalent to the contributions of all its $m(m-1)/2$ effective component pairs $(T_i, T_j)_{h_1 \dots h_r}$, then each $S_p(h_1 \dots h_r)$ adds a value $4n_p(h_1 \dots h_r)(m - n_p(h_1 \dots h_r))$ to c_{qq}^* . This value is maximum when (i) $n_p(h_1 \dots h_r) = m/2$ (for m even) and (ii) $n_p(h_1 \dots h_r) = (m-1)/2$ or $n_p(h_1 \dots h_r) = (m+1)/2$ (for m odd). Since there are NQ such effective choice sets $S_p(h_1 \dots h_r)$ in d , then we have the required expression for $\max(\text{trace}(C))$. \square

From Lemma 2.2 and Lemma 2.3, it follows that a design $d \in \mathcal{D}_{N,n,m}$ is universally optimal for F , if (i) C is diagonal, i.e., $\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ = \eta_{h_1 \dots h_r, k_1 \dots k_l}^-$, for all $h_1 \dots h_r, k_1 \dots k_l \in N_F$ and (ii) $\text{trace}(C)$ is maximum, i.e., $n_p(h_1 \dots h_r) = m/2$ (m even) and $n_p(h_1 \dots h_r) = (m-1)/2$ or $(m+1)/2$ (m odd), for all $S_p(h_1 \dots h_r)$, $p = 1, \dots, N$, $h_1 \dots h_r \in N_F$. Henceforth in this paper, by optimal design, we mean universally optimal choice design.

3. Optimal designs for the main effects and the broader main effects models

Let $F = \{F_{h_1} : h_1 = 1, \dots, n\}$ be the set of all main effects of our interest. Let $F_{(2)} = \{F_{h_1 h_2} : h_1 < h_2, h_1, h_2 = 1, \dots, n\}$ be the set of all two factor interaction effects and $N_{F(2)} = \{h_1 h_2 : F_{h_1 h_2} \in F_{(2)}\}$. Under the main effects model we obtain optimal designs for F when two and higher order interactions effects are assumed to be zero and under the broader main effects we obtain optimal designs for F when three and higher order interaction effects are assumed to be zero. Let $B_{(1)}$ be the contrast matrix corresponding to all the main effects and $B_{(2)}$ be the contrast matrix corresponding to all the two factor interaction effects. Then for a design $d \in \mathcal{D}_{N,n,m}$, the information matrix C ($= C_{(1)}$ say) of F under the main effects model is

$$C_{(1)} = (1/2^n) B_{(1)} \Lambda B'_{(1)},$$

and the information matrix C ($= C_{(2)}$ say) of F under the broader main effects model is

$$C_{(2)} = (1/2^n) \{B_{(1)} \Lambda B'_{(1)} - B_{(1)} \Lambda B'_{(2)} [B_{(2)} \Lambda B'_{(2)}]^{-1} B_{(2)} \Lambda B'_{(1)}\}.$$

Note that $B_{(1)} \Lambda B'_{(2)} [B_{(2)} \Lambda B'_{(2)}]^{-1} B_{(2)} \Lambda B'_{(1)}$ is a non-negative definite matrix and

$$\text{trace}(2^n C_{(2)}) = \text{trace}(B_{(1)} \Lambda B'_{(1)}) - \text{trace}(B_{(1)} \Lambda B'_{(2)} [B_{(2)} \Lambda B'_{(2)}]^{-1} B_{(2)} \Lambda B'_{(1)}).$$

Thus $\text{trace}(C_{(2)}) \leq \text{trace}(C_{(1)})$ with equality attaining when $B_{(1)} \Lambda B'_{(2)}$ is a null matrix. Following Lemma 2.2 and Lemma 2.3, we see that a design d is optimal for estimating F under the main effects model if (i) $\eta_{h_1, k_1}^+ = \eta_{h_1, k_1}^-$, for all $h_1, k_1 \in N_F$ and (ii) $n_{p(h_1)} = m/2$ (m even) and $n_{p(h_1)} = (m-1)/2$ or $(m+1)/2$ (m odd), for all $S_p(h_1)$, $p = 1, \dots, N$, $h_1 \in N_F$. A design d is optimal for estimating F under the broader main effects model if it satisfies the above two conditions along with (iii) $\eta_{h_1, k_1 k_2}^+ = \eta_{h_1, k_1 k_2}^-$, for all $h_1 \in N_F$, $k_1 k_2 \in N_{F(2)}$. Therefore if a design d is optimal for estimating F under the broader main effects model then it is also optimal under the main effects model but the converse is not always true. In what follows, in this section, we first obtain the optimal designs for F under the broader main effects model and as a corollary we obtain the optimal design for F under the main effects model.

We use generator technique to construction optimal designs in this section and in the next section. Let $d = (A_1, \dots, A_m)$ be a design in $\mathcal{D}_{N,n,m}$ and $g_j = (g_{j1}, \dots, g_{jn})$ be the j -th generator, where $g_{jr} = 0, 1$. When A_i is generated from A_1 using g_j , then it is denoted by $A_i = A_1 + g_j$, and is defined as $A_i(p, r) = A_1(p, r) + g_{jr} \pmod{2}$, $1 \leq p \leq N$, $1 \leq r \leq n$.

Let \bar{A}_i denotes the complement of A_i , i.e., the elements 0 and 1 interchange their respective positions in A_i and $\bar{d} = (\bar{A}_1, \dots, \bar{A}_m)$ denotes the complement design of $d = (A_1, \dots, A_m)$. Similarly, let \bar{T}_i and \bar{g}_i are the complements of T_i and g_i . We now obtain some optimal designs for given n and m .

Theorem 3.1. *Let ν and $\nu' (< \nu)$ are two conjugative numbers such that Hadamard matrices of order ν and ν' exist. Let $G = \{g_1, \dots, g_\alpha\}$ be a set of α different generators such that both g_i and $\bar{g}_i \notin G$. Then for $m = \text{even}$, there exists a design d_1^* in $\mathcal{D}_{\nu,n,m}$, and for $m = \text{odd}$, there exists a design d_2^* in $\mathcal{D}_{2\nu,n,m}$, which are optimal for estimating F under the broader main effects model, where $\nu' < n \leq \nu$, $m = 2, \dots, 2\alpha + 1, 2\alpha + 2$.*

Proof. Let $d = (A_1, \dots, A_m)$. Consider H be a Hadamard matrix of order ν . Let A_1 is derived from H by taking any n columns of H and replacing all the -1 entries with 0 entries. Let $A_2 = \bar{A}_1$. Using the α generators from G , generate the other components of d in the following manner

$$A_{2u+1} = A_1 + g_u, A_{2u+2} = A_2 + g_u, \quad u = 1, \dots, \alpha.$$

Consider, $d_1^* = d$ (m even) and $d_2^* = \{d, \bar{d}\}$ (m odd).

Claim: When $m = \text{even}$, d_1^* is optimal in $\mathcal{D}_{\nu,n,m}$ for estimating F under the broader main effects model.

Note that the information matrix C of the design d is the sum of information matrix corresponding to all component paired design $\delta_{ij} = (A_i, A_j)$, $i < j$, $i, j = 1, \dots, m$. Note also that both δ_{ij} and $\bar{\delta}_{ij}$ are present in d_1^* . Now according to the construction of d_1^* , we see that both the effective component pairs $(00, 11)_{h_1, k_1}$ and $(01, 10)_{h_1, k_1}$ occur equally often in each δ_{ij} of d_1^* . We also observe that the existence of the effective component pair $(00, 11)_{h_1, k_1 k_2}$ in δ_{ij} ensures the existence of $(01, 10)_{h_1, k_1 k_2}$ in the corresponding effective component pair of $\bar{\delta}_{ij}$ and vice-versa. Thus for the design d_1^* , $\eta_{h_1, k_1}^+ = \eta_{h_1, k_1}^-$, for all $h_1, k_1 \in N_F$, and $\eta_{h_1, k_1 k_2}^+ = \eta_{h_1, k_1 k_2}^-$, for all $h_1 \in N_F$, $k_1 k_2 \in N_{F(2)}$. Hence C is a diagonal matrix with $B_{(1)} \Lambda B'_{(2)} = 0$. Also note that for every $S_p(h_1)$ of d_1^* , $n_p(h_1) = m/2$, $p = 1, \dots, \nu$, $h_1 \in N_F$, and thus $\text{trace}(C)$ is maximum. Hence d_1^* is optimal in $\mathcal{D}_{\nu,n,m}$ for estimating F under the broader main effects model.

Claim: When $m = \text{odd}$, d_2^* is optimal in $\mathcal{D}_{2\nu,n,m}$ for estimating F under the broader main effects model.

The proof follows in the same way as $m = \text{even}$ case on noting that for every component paired design δ_{ij} of d , the corresponding component paired design in \bar{d} is $\bar{\delta}_{ij}$ and for every $S_p(h_1)$ of d_2^* , $n_p(h_1) = (m-1)/2$ or $(m+1)/2$, $p = 1, \dots, 2\nu$, $h_1 \in N_F$. \square

Corollary 3.1. *For any odd or even m , both $d = (A_1, \dots, A_m)$ and $\bar{d} = (\bar{A}_1, \dots, \bar{A}_m)$ are optimal in $\mathcal{D}_{\nu,n,m}$ for estimating F under the main effects model.*

Example 3.1. *Suppose $F = \{F_1, \dots, F_8\}$ and we want to construct optimal designs for $m = 5$ and $m = 6$ under the broader main effects model. Take H be the Hadamard matrix of order 8 and A be the matrix is generated from H by replacing all its -1 entries with 0 entries. For $m = 6$, let $d_1 = (A_1, A_2, A_3, A_4, A_5, A_6)$, where $A_1 = A$ and $A_2 = \bar{A}$, $A_3 = A_1 + g_1$, $A_4 = A_2 + g_1$, $A_5 = A_1 + g_2$ and $A_6 = A_2 + g_2$. If we take $g_1 = (11100000)$ and $g_2 = (00000011)$, then*

$$d_1 = \begin{pmatrix} 11111111, & 00000000, & 00011111, & 11100000, & 11111100, & 00000011 \\ 10101010, & 01010101, & 01001010, & 10110101, & 10101001, & 01010110 \\ 11001100, & 00110011, & 00101100, & 11010011, & 11001111, & 00110000 \\ 10011001, & 01100110, & 01111001, & 10000110, & 10011010, & 01100101 \\ 11110000, & 00001111, & 00010000, & 11101111, & 11110011, & 00001100 \\ 10100101, & 01011010, & 01000101, & 10111010, & 10100110, & 01011001 \\ 11000011, & 00111100, & 00100011, & 11011100, & 11000000, & 00111111 \\ 10010110, & 01101001, & 01110110, & 10001001, & 10010101, & 01101010 \end{pmatrix}$$

is optimal in $\mathcal{D}_{8,8,6}$ for F under the broader main effects model.

Now for $m = 5$, let $d_2 = (A_1, A_2, A_3, A_4, A_5)$. Then $d_2^* = \{d_2, \bar{d}_2\}$ is optimal in $\mathcal{D}_{16,8,5}$ under the broader main effects model. Note that d_1 and d_2 are also optimal in $\mathcal{D}_{8,8,6}$ and $\mathcal{D}_{8,8,5}$ respectively for F under the main effects model.

The construction of Theorem 3.1 is a general construction in a sense that for given m and n , one can always find an optimal design for some N ($\geq n$), provided a Hadamard matrix of order N exists. We now construct optimal designs for some specific values of m and for any n , which are better than the designs provided by Theorem 3.1 in a sense that they produce optimal designs in less number of choice sets ($N < n$, in most of the cases). For example, let ν be the least number greater than or equal to n such that a Hadamard matrix of order ν exists. Let H be a normalized Hadamard matrix of order ν and A be the matrix is derived from H by taking only n columns of H and replacing all the -1 entries with 0 entries. Let $A = (T_1, \dots, T_\nu)'$, where T_i is a typical treatment combination. Then it is easy to see that $d_1^* = (T_1, \dots, T_\nu, \bar{T}_1, \dots, \bar{T}_\nu)$ is optimal in $\mathcal{D}_{1,n,2\nu}$ for estimating F under the broader main effects model. Now if the above A does not contains the first column of H , then $d_2^* = \{(T_1, \dots, T_\nu), (\bar{T}_1, \dots, \bar{T}_\nu)\}$ is optimal in $\mathcal{D}_{2,n,\nu}$, $n < \nu$, for estimating F under the broader main effects model. Note that the choice sets (T_1, \dots, T_ν) or $(\bar{T}_1, \dots, \bar{T}_\nu)$ is optimal in $\mathcal{D}_{1,n,\nu}$, $n < \nu$, for estimating F under the main effects model.

Example 3.2. Let $H = (1111, 1010, 1100, 1001)'$ be a normalized Hadamard matrix of order 4, then $d_1^* = (1111, 1010, 1100, 1001, 0000, 0101, 0011, 0110)$ is optimal in $\mathcal{D}_{1,4,8}$ for $F = \{F_1, F_2, F_3, F_4\}$ and $d_2^* = \{(111, 100, 010, 001), (000, 011, 101, 110)\}$ is optimal in $\mathcal{D}_{2,3,4}$ for $F = \{F_1, F_2, F_3\}$ under the broader main effects model.

From the discussion above we see that for given $m = \nu$, the two-run design d_2^* is optimal for $n < \nu$. We now generalize this idea for the cases when $n \geq \nu$. Before presenting the next construction, we need to define a new operation. Let $S_1 = (T_{11}, \dots, T_{1m})$ and $S_2 = (T_{21}, \dots, T_{2m})$ are two choice sets of size m with n_1 and n_2 factors respectively. We denote the direct addition of S_1 and S_2 as $S_1 \oplus S_2$ and is defined by $S_1 \oplus S_2 = (T_{11}T_{21}, \dots, T_{1m}T_{2m})$, which is a new choice sets of size m with $n_1 + n_2$ factors. The definition carry forward for two choice designs in the similar way. Suppose $d_1 = \{S_{11}, \dots, S_{1N}\}$ and $d_2 = \{S_{21}, \dots, S_{2N}\}$ are two designs with N choice sets with n_1 and n_2 factors respectively. Then the direct addition of d_1 and d_2 is denoted by $d_1 \oplus d_2$ and is defined by $d_1 \oplus d_2 = \{S_{11} \oplus S_{21}, \dots, S_{1N} \oplus S_{2N}\}$, which is a new design with N choice sets with $n_1 + n_2$ factors.

Theorem 3.2. Let H be a Hadamard matrix of order ν . Then there exists an optimal design d^* in $\mathcal{D}_{2^{\alpha+1},n,\nu}$ for estimating F under the broader main effects model, $2^{\alpha-1}(\nu - 1) < n \leq 2^\alpha(\nu - 1)$, $\alpha = 1, 2, 3, \dots$

Proof. Let H be a normalized Hadamard matrix of order ν and A be the $(\nu - 1) \times \nu$ matrix is derived from H by deleting the 1-st column and replacing all the -1 entries with 0 entries. Let d_0 is the choice set of size ν whose treatments are the rows of A . Then for $\alpha = 1$, let

$$d_1 = \{d_0 \oplus d_0, d_0 \oplus \bar{d}_0\}.$$

Similarly, for any $\alpha > 1$, $d_\alpha = \{d_{\alpha-1} \oplus d_{\alpha-1}, d_{\alpha-1} \oplus \bar{d}_{\alpha-1}\}.$

Let d be the design with 2^α choice sets is generated from d_α by taking only n corresponding factors from each treatment of each choice set of d_α . Let $d^* = \{d, \bar{d}\}$. Thus d^* is a design with $2^{\alpha+1}$ choice sets and n factors, $2^{\alpha-1}(\nu - 1) < n \leq 2^\alpha(\nu - 1)$.

Claim: d^* is optimal in $\mathcal{D}_{2^{\alpha+1}, n, \nu}$ for estimating F under the broader main effects model.

Note from the construction of d (or \bar{d}) that both the effective component pairs $(00, 11)_{h_1, k_1}$ and $(01, 10)_{h_1, k_1}$ occur equally often in d (or \bar{d}). Thus $\eta_{h_1, k_1}^+ = \eta_{h_1, k_1}^-$, for all $h_1, k_1 \in N_F$. We also observe that the existence of a effective component pair $(00, 11)_{h_1, k_1 k_2}$ in d ensures the existence of $(01, 10)_{h_1, k_1 k_2}$ in the corresponding effective component pair of \bar{d} and vice-versa. Thus $\eta_{h_1, k_1 k_2}^+ = \eta_{h_1, k_1 k_2}^-$, for all $h_1 \in N_F, k_1 k_2 \in N_{F(2)}$. Also note that for every $S_p(h_1)$ of d^* , $n_p(h_1) = \nu/2$, $p = 1, \dots, 2^{\alpha+1}$, $h_1 \in N_F$. Hence d^* is optimal in $\mathcal{D}_{2^{\alpha+1}, n, \nu}$ for estimating F under the broader main effects model. \square

Corollary 3.2. *Note that both d and \bar{d} are optimal in $\mathcal{D}_{2^\alpha, n, \nu}$ for estimating F under the main effects model, $2^{\alpha-1}(\nu - 1) < n \leq 2^\alpha(\nu - 1)$, $\alpha = 1, 2, 3, \dots$*

Example 3.3. *Suppose we want an optimal design for $F = \{F_1, F_2, F_3, F_4, F_5\}$ and $m = 4$. Let $H = (1111, 1100, 1010, 1001)'$ be a normalized Hadamard matrix of order 4, then $d_0 = \{111, 100, 010, 001\}$. Therefore for $\alpha = 1$
 $d_1 = \{d_0 \oplus d_0, d_0 \oplus \bar{d}_0\} = \{(111111, 100100, 001001, 010010), (111000, 100011, 001110, 010101)\}.$
Let d be the design is obtained from d_1 removing the last factor of each treatments, then $d^* = \{d, \bar{d}\} = \{(11111, 10010, 00100, 01001), (11100, 10001, 00111, 01010), (00000, 01101, 11011, 10110), (00011, 01110, 11000, 10101)\}$ is optimal in $\mathcal{D}_{4, 5, 4}$ for F under the broader main effects model. Also note that $d = \{(11111, 10010, 00100, 01001), (11100, 10001, 00111, 01010)\}$ or $\bar{d} = \{(00000, 01101, 11011, 10110), (00011, 01110, 11000, 10101)\}$ are optimal in $\mathcal{D}_{2, 5, 4}$ for F under the main effects model.*

4. Optimal designs for the main effects and the specified interaction effects model

In many choice investigation problems researchers need the information about two and higher order interaction effects along with the main effects. In most of such cases, the information about all the interaction effects are not so important but interaction effects with some specified factors are much more important. In this section we obtain optimal designs for such situations. We first present constructions of optimal designs for estimating all the main effects and all the interaction effects with a single specified factor and later on we generalize this idea with more than one specified factors. Without loss of generality we assume the first factor to be the specified factor. Thus if F is the set of all factorial effects of our interest, then $F = \{F_1, \dots, F_n, F_{12}, \dots, F_{1n}, F_{123}, \dots, F_{12\dots n}\}$. The factorial effects which are not in F are assumed to be zero in this section. For a component matrix A_i of a design $d = (A_1, \dots, A_i, \dots, A_m)$, we define $\mathcal{C}_{A_i}(h_1 \dots h_r) = (i_{h_1 \dots h_r}^*)_{N \times 1}$ to be the effective column of A_i corresponding to the factorial effect $F_{h_1 \dots h_r}$. Let H is a Hadamard matrix of order 2^α , $\alpha \geq 2$, where

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (4.1)$$

Here \otimes denotes the Kronecker product. Let

$$A = \{a \text{ (0,1) matrix is derived from H of (4.1) after replacing all the } -1 \text{ by } 0\}. \quad (4.2)$$

Note that every effective column $\mathcal{C}_A(h_1 \dots h_r)$ is equivalent to a column of A . We use this matrix A in many of our constructions of optimal designs in this section.

Theorem 4.1. *For $2^{\alpha-1} < n \leq 2^\alpha$, $\alpha \geq 2$, there exists an optimal design d^* in $\mathcal{D}_{2^\alpha, n, 4}$ for estimating F .*

Proof. Let A_1 is a $2^\alpha \times n$ matrix is derived from (4.2) by taking any n columns of A (including the first column), $2^{\alpha-1} < n \leq 2^\alpha$, $\alpha \geq 2$. Consider $d^* = (A_1, A_2, A_3, A_4)$, where $A_2 = \bar{A}_1$, $A_3 = A_1 + g$ and $A_4 = A_2 + g$, with $g = (100\dots 0)$.

Claim: d^* is optimal in $\mathcal{D}_{2^\alpha, n, 4}$ for estimating F .

Note that every effective column $\mathcal{C}_{A_1}(h_1 \dots h_r)$ changes twice in d^* . Therefore $n_p(h_1 \dots h_r) = 2$, for every $S_p(h_1 \dots h_r)$ in d^* , $p = 1, \dots, 2^\alpha$, $h_1 \dots h_r \in N_F$, and hence $\text{trace}(C)$ is maximum.

Now we only need to show that C is diagonal. Let $w_1 = i_{h_1 \dots h_r}^*$ and $w_2 = i_{k_1 \dots k_l}^*$ are the p -th elements of $\mathcal{C}_{A_1}(h_1 \dots h_r)$ and $\mathcal{C}_{A_1}(k_1 \dots k_l)$ respectively. Since every effective column of A_1 changes exactly two times in d^* , then each effective choice set $S_p(h_1 \dots h_r, k_1 \dots k_l)$ of d^* is any of the following two types

$$\text{type-1: } S_p(h_1 \dots h_r, k_1 \dots k_l) \equiv (w_1 w_2, w_1 \bar{w}_2, \bar{w}_1 w_2, \bar{w}_1 \bar{w}_2)_{h_1 \dots h_r, k_1 \dots k_l}$$

$$\text{type-2: } S_p(h_1 \dots h_r, k_1 \dots k_l) \equiv (w_1 w_2, \bar{w}_1 \bar{w}_2, w_1 \bar{w}_2, \bar{w}_1 w_2)_{h_1 \dots h_r, k_1 \dots k_l}$$

Note that if $S_p(h_1 \dots h_r, k_1 \dots k_l)$ is of *type-1*, then $(00, 11)_{h_1 \dots h_r, k_1 \dots k_l}$ occurs twice when $w_1 = w_2$ and $(01, 10)_{h_1 \dots h_r, k_1 \dots k_l}$ occurs twice when $w_1 \neq w_2$ in $S_p(h_1 \dots h_r, k_1 \dots k_l)$. Similarly, if $S_p(h_1 \dots h_r, k_1 \dots k_l)$ is of *type-2*, then both $(00, 11)_{h_1 \dots h_r, k_1 \dots k_l}$ and $(01, 10)_{h_1 \dots h_r, k_1 \dots k_l}$ occur once each in $S_p(h_1 \dots h_r, k_1 \dots k_l)$. Now we have the following two cases

Case-1: If $\mathcal{C}_{A_1}(h_1 \dots h_r)$ and $\mathcal{C}_{A_1}(k_1 \dots k_l)$ are different, then there are equal number of effective choice sets $S_p(h_1 \dots h_r, k_1 \dots k_l)$ in d^* for which $w_1 = w_2$ and $w_1 \neq w_2$. Therefore whether it is of *type-1* or *type-2*, $\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ = \eta_{h_1 \dots h_r, k_1 \dots k_l}^-$, for all $h_1 \dots h_r, k_1 \dots k_l \in N_F$.

Case-2: If $\mathcal{C}_{A_1}(h_1 \dots h_r)$ and $\mathcal{C}_{A_1}(k_1 \dots k_l)$ are same, then $S_p(h_1 \dots h_r, k_1 \dots k_l)$ is of *type-2*. Therefore $\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ = \eta_{h_1 \dots h_r, k_1 \dots k_l}^-$, for all $h_1 \dots h_r, k_1 \dots k_l \in N_F$.

Thus C is a equal diagonal matrix with maximum *trace* and hence d^* is optimal in $\mathcal{D}_{2^\alpha, n, 4}$ for estimating F . \square

Corollary 4.1. *Let $F^* \subset F$, then d^* is also optimal in $\mathcal{D}_{2^\alpha, n, 4}$ for estimating F^* .*

Theorem 4.2. *For $2^{\alpha-1} < n \leq 2^\alpha$, $\alpha \geq 2$, there exists an optimal design d^* in $\mathcal{D}_{2^{\alpha+1}, n, 3}$ for estimating F .*

Proof. Let A_1 is a $2^\alpha \times n$ matrix is derived from (4.2) by taking any n columns of A (including the first column), $2^{\alpha-1} < n \leq 2^\alpha$. Let $d = (A_1, A_2, A_3)$, where $A_2 = \bar{A}_1$, $A_3 = A_1 + g$ with $g = (100\dots 0)$. Consider $d^* = \{d, \bar{d}\}$.

Claim: d^* is optimal in $\mathcal{D}_{2^{\alpha+1}, n, 3}$ for estimating F .

Note that every effective column $\mathcal{C}_{A_1}(h_1 \dots h_r)$ changes either once or twice in d^* . Therefore $n_p(h_1 \dots h_r) = 1$ or $n_p(h_1 \dots h_r) = 2$, for every $S_p(h_1 \dots h_r)$ in d^* , $p = 1, \dots, 2^{\alpha+1}$, $h_1 \dots h_r \in N_F$, and hence $\text{trace}(C)$ is maximum.

Now we only need to show that C is diagonal. Let $w_1 = i_{h_1 \dots h_r}^*$ and $w_2 = i_{k_1 \dots k_l}^*$ are the p -th elements of $\mathcal{C}_{A_1}(h_1 \dots h_r)$ and $\mathcal{C}_{A_1}(k_1 \dots k_l)$ respectively. Since every effective column of A_1 changes either once or twice in d^* , then each effective choice set $S_p(h_1 \dots h_r, k_1 \dots k_l)$ of d^* is any of the following types

When both $\mathcal{C}_{A_1}(h_1 \dots h_r)$ and $\mathcal{C}_{A_1}(k_1 \dots k_l)$ change once

$$\text{type-1: } S_p(h_1 \dots h_r, k_1 \dots k_l) \equiv (w_1 w_2, w_1 w_2, \bar{w}_1 \bar{w}_2)_{h_1 \dots h_r, k_1 \dots k_l}$$

$$\text{type-2: } S_p(h_1 \dots h_r, k_1 \dots k_l) \equiv (w_1 w_2, w_1 \bar{w}_2, \bar{w}_1 w_2)_{h_1 \dots h_r, k_1 \dots k_l}$$

When $\mathcal{C}_{A_1}(h_1 \dots h_r)$ changes once and $\mathcal{C}_{A_1}(k_1 \dots k_l)$ changes twice or vice-versa

$$\text{type-3: } S_p(h_1 \dots h_r, k_1 \dots k_l) \equiv (w_1 w_2, w_1 \bar{w}_2, \bar{w}_1 \bar{w}_2)_{h_1 \dots h_r, k_1 \dots k_l}$$

$$\text{type-4: } S_p(h_1 \dots h_r, k_1 \dots k_l) \equiv (w_1 w_2, \bar{w}_1 w_2, \bar{w}_1 \bar{w}_2)_{h_1 \dots h_r, k_1 \dots k_l}$$

When both $\mathcal{C}_{A_1}(h_1 \dots h_r)$ and $\mathcal{C}_{A_1}(k_1 \dots k_l)$ change twice

$$\text{type-5: } S_p(h_1 \dots h_r, k_1 \dots k_l) \equiv (w_1 w_2, \bar{w}_1 \bar{w}_2, \bar{w}_1 \bar{w}_2)_{h_1 \dots h_r, k_1 \dots k_l}$$

Now we have the following two cases.

Case-1: If $\mathcal{C}_{A_1}^{h_1 \dots h_r}$ and $\mathcal{C}_{A_1}^{k_1 \dots k_l}$ are different, then there are equal number of effective choice sets $S_p(h_1 \dots h_r, k_1 \dots k_l)$ in d^* , for which $w_1 = w_2$ and $w_1 \neq w_2$. From any of the above five types we see that if an effective choice set with $w_1 = w_2$ contains $(00, 11)_{h_1 \dots h_r, k_1 \dots k_l}$, then an effective choice set with $w_1 \neq w_2$ contains $(01, 10)_{h_1 \dots h_r, k_1 \dots k_l}$.

Case-2: If $\mathcal{C}_{A_1}^{h_1 \dots h_r}$ and $\mathcal{C}_{A_1}^{k_1 \dots k_l}$ are same, then every effective choice set $S_p(h_1 \dots h_r, k_1 \dots k_l)$ of d^* is either of *type-2* or *type-3* or *type-4*. Note that if an effective choice set in d is of *type-2*, then the corresponding choice set of \bar{d} is of *type-3* or *type-4* and vice-versa. Thus if an effective choice set of d contains $(00, 11)_{h_1 \dots h_r, k_1 \dots k_l}$, then the corresponding effective choice set of \bar{d} contains $(01, 10)_{h_1 \dots h_r, k_1 \dots k_l}$ and vice-versa.

Considering both the above cases we see that $\eta_{h_1 \dots h_r, k_1 \dots k_l}^+ = \eta_{h_1 \dots h_r, k_1 \dots k_l}^-$, for all $h_1 \dots h_r, k_1 \dots k_l \in N_F$. Thus C is a equal diagonal matrix with maximum *trace* and hence d^* is optimal in $\mathcal{D}_{2^{\alpha+1}, n, 3}$ for estimating F . \square

Corollary 4.2. *Let $F^* \subset F$, then d^* is also optimal in $\mathcal{D}_{2^{\alpha+1}, n, 3}$ for estimating F^* .*

The results of Corollary 4.1 and Corollary 4.2 can be further improved in terms of lesser number of choice sets if F^* contains all the main effects and all the specified two factor interaction effects with one factor, i.e., $F^* = \{F_1, \dots, F_n, F_1 F_2, \dots, F_1 F_n\}$.

Theorem 4.3. *For given n , let ν be the least number such that a Hadamard matrix of order ν exists. Then there exists an optimal design d^* in $\mathcal{D}_{\nu, n, 4}$ for estimating F^* .*

Proof. The proof follows same way as of Theorem 4.1, by taking H of (4.1) to be a normalized Hadamard matrix of order ν . \square

Theorem 4.4. *For given n , let ν be the least number such that a Hadamard matrix of order ν exists. Then there exists an optimal design d^* in $\mathcal{D}_{\nu, n, 3}$ for estimating F^* .*

Proof. The proof follows same way as of Theorem 4.2, by taking H of (4.1) to be a normalized Hadamard matrix of order ν . \square

Example 4.1. Suppose we have $n = 4$ factors and we want an optimal design for $F = \{F_1, F_2, F_3, F_4, F_{12}, F_{13}, F_{14}, F_{123}, F_{134}, F_{1234}\}$ or any subset F^* of F . Then

$$d^* = \begin{pmatrix} (1111, & 0000, & 0111, & 1000) \\ (1010, & 0101, & 0010, & 1101) \\ (1100, & 0011, & 0100, & 1011) \\ (1001, & 0110, & 0001, & 1110) \end{pmatrix}$$

is optimal in $\mathcal{D}_{4,4,4}$ for estimating F or any subset F^* of F .

So far we discuss about the optimal designs for estimating main effects and specified interaction effects with one factor. We now generalized the idea to get optimal designs for estimating main effects and specified interaction effects with more than one factor. Suppose we have a partition of factors into two groups, say, $\mathbf{f}_1 = \{f_{h_1}, \dots, f_{h_r}\}$ and $\mathbf{f}_2 = \{f_{k_1}, \dots, f_{k_l}\}$ and we are interested in estimating all the main effects and all the specific interaction effects between each factor of the group \mathbf{f}_1 to all the factors of the group \mathbf{f}_2 . Therefore the set of all factorial effects of interests are $F = \{F_{h_1}, \dots, F_{h_r}, F_{k_1}, \dots, F_{k_l}, F_{h_1k_1}, \dots, F_{h_1k_1\dots k_l}, F_{h_2k_1}, \dots, F_{h_rk_1\dots k_l}\}$.

Theorem 4.5. for $2^{\alpha-1} < n \leq 2^\alpha$, $\alpha \geq 2$, there exists an optimal design d^* in $\mathcal{D}_{2^\alpha, n, 4}$ for estimating F .

Proof. Let A_1 is a $2^\alpha \times n$ matrix is derived from (4.2) by taking any n columns of A (including the first column), $2^{\alpha-1} < n \leq 2^\alpha$, $\alpha \geq 2$. Consider $d^* = (A_1, A_2, A_3, A_4)$, where $A_2 = \bar{A}_1$, $A_3 = A_1 + g$ and $A_4 = A_2 + g$. Here g is a generator whose first r elements are 1, i.e., $g = (11\dots 10\dots 0)$. The rest of the proof follows in the similar way as Theorem 4.1. \square

Corollary 4.3. Let $F^* \subset F$, then d^* is also optimal in $\mathcal{D}_{2^\alpha, n, 4}$ for estimating F^* .

Theorem 4.6. for $2^{\alpha-1} < n \leq 2^\alpha$, $\alpha \geq 2$, there exists an optimal design d^* in $\mathcal{D}_{2^{\alpha+1}, n, 3}$ for estimating F .

Proof. Let A_1 is a $2^\alpha \times n$ matrix is derived from (4.2) by taking any n columns of A (including the first column), $2^{\alpha-1} < n \leq 2^\alpha$. Let $d = (A_1, A_2, A_3)$, where $A_2 = \bar{A}_1$, $A_3 = A_1 + g$. Here g is a generator whose first r elements are 1, i.e., $g = (11\dots 10\dots 0)$. Consider $d^* = \{d, \bar{d}\}$. The rest of the proof follows same way as Theorem 4.2. \square

Corollary 4.4. Let $F^* \subset F$, then d^* is also optimal in $\mathcal{D}_{2^{\alpha+1}, n, 3}$ for estimating F^* .

Example 4.2. Suppose we have $n = 4$ factors and we want an optimal design for $F = \{F_1, F_2, F_3, F_4, F_{13}, F_{14}, F_{23}, F_{24}, F_{134}, F_{234}\}$ or any subset F^* of F . Then

$$d^* = \begin{pmatrix} (1111, & 0000, & 0011, & 1100) \\ (1010, & 0101, & 0110, & 1001) \\ (1100, & 0011, & 0000, & 1111) \\ (1001, & 0110, & 0101, & 1010) \end{pmatrix}$$

is optimal in $\mathcal{D}_{4,4,4}$ for estimating F or any subset F^* of F .

Table 4.1: Number of choice sets (N) required for the optimal designs for given m and n .

Main effects model											
m \ n	2	3	4	5	6	7	8	9	10	11	12
2	2	4	4	8	8	8	8	12	12	12	12
3	2	4	4	8	8	8	8	12	12	12	12
4	1	1	2	2	2	4	4	4	4	4	4
5		4	4	8	8	8	8	12	12	12	12
6		4	4	8	8	8	8	12	12	12	12
7		4	4	8	8	8	8	12	12	12	12
8		1	1	1	1	1	2	2	2	2	2
Broader main effects model											
2	2	4	4	8	8	8	8	12	12	12	12
3	2	8	8	16	16	16	16	24	24	24	24
4	1	2	4	4	4	8	8	8	8	8	8
5		8	8	16	16	16	16	24	24	24	24
6		4	4	8	8	8	8	12	12	12	12
7		8	8	16	16	16	16	24	24	24	24
8		1	1	2	2	2	4	4	4	4	4
Main plus specified two factor interaction effects model											
3	4	8	8	16	16	16	16	24	24	24	24
4		4	4	8	8	8	8	12	12	12	12
Main plus all specified interaction effects model											
3	4	8	8	16	16	16	16	32	32	32	32
4		4	4	8	8	8	8	16	16	16	16

5. Discussion

In this paper, we have obtained optimal choice designs for some broader class of model set up than the existing ones in the literature. The constructions are very easy and simple, yet the optimal designs are obtained in practical number of choice sets. From the Table 4.1, it is seen that when $m = 4t$, $t = 1, 2, \dots$, one gets optimal designs in least number of choice sets than any other m for each of the model set up. When $m = 4t + 2$, $t = 0, 1, \dots$, optimal designs for the broader main effects model are obtained in same number of choice sets as the main effects model but when m is odd it takes double.

Considering the fact that all the two or higher order interactions effects are not equally important in any choice investigation problem, the designs present in this paper are quite useful to reduce the cognitive burden of the respondents. For example, in a 2^4 choice investigation problem, for an optimal design, one needs 80 choice sets of size 2 to estimate all the main effects and all the two factor interaction effects (Street and Burgess (2004)), whereas one needs only 4 choice sets of size 4 to estimate all the main effects and all the two and higher order specified interaction effects.

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